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THE GROUP $G_1(R\pi)$ FOR π A FINITE ABELIAN GROUP

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1. Introduction

For a right noetherian ring R, $G_0(R)$ denotes the Grothendieck group $K_0(\mathcal{M}_R)$ of the category $\mathcal{M}_R$ of finitely generated right R-modules.

Let π be a finite abelian group and $R\pi$ its group ring over R. Then H. Lenstra [2] has obtained a beautiful calculation of $G_0(R\pi)$. It is natural to ask whether Lenstra's formula generalizes to the higher K-groups of the category $\mathcal{M}_R$. Unfortunately this does not seem to be the case. Nevertheless Lenstra's formula does generalize to the group $G_1(R\pi)$ (of [1, p. 453], which does not coincide with Quillen's $K_1(\mathcal{M}_{R\pi})$). We present here this calculation of $G_1(R\pi)$, following Lenstra rather closely.

Let $X(\pi)$ denote the set of cyclic quotient groups of π. If $\varrho \in X(\pi)$ has order n and a generator t we put

$$R(\varrho) = R\varrho/\phi_n(t)R\varrho$$

where ϕ_n denotes the nth cyclotomic polynomial; the two-sided ideal $\Phi_n(t)R_\varrho$ does not depend on the choice of the generator t (cf. [2]). The main result is an isomorphism of the form

$$G_i(R\pi) \cong \bigoplus_{\varrho \in X(\pi)} G_i(R(\varrho))/H_\varrho, \tag{1.1}$$

where H_ϱ will be described below. This isomorphism is Lenstra's result for $i=0$, and we prove it here for $i=1$.

Recall that, for any right noetherian ring R, the abelian group $G_0(R)$ is presented by generators $[M]$ for $M \in \mathcal{M}_R$ and relations $[M]=[M']+[M'']$ for each exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{M}_R$.

Similarly $G_1(R)$ is presented by generators $[M, \alpha]$, $M \in \mathcal{M}_R$, $\alpha \in \mathrm{Aut}_R(M)$, and exact sequence relations as above, plus relations $[M, \alpha\beta] = [M, \alpha] + [M, \beta]$ for $\alpha, \beta \in \mathrm{Aut}_R(M)$.

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Now let π be a finite abelian group as above. If $\varrho \in X(\pi)$ has order n then $\mathcal{M}_{R(\varrho)/nR(\varrho)}$ can be considered to be a subcategory of $\mathcal{M}_{R(\varrho)}$, whence a homomorphism

$$G_i(R(\varrho)/nR(\varrho)) \to G_i(R(\varrho)) \quad (i=0,1).$$

The image of this homomorphism is the group H_ϱ in (1.1) above. Thus $G_0(R(\varrho))/H_\varrho$ (resp. $G_1(R(\varrho))/H_\varrho$) is presented by adding to the defining relations above the additional relations $[M]=0$ (resp. $[M,\alpha]=0$) whenever $n \cdot M = 0$.

If we put $R\langle\varrho\rangle = R(\varrho)[1/n]$ then there is an exact localization sequence [1, p. 492]

$$G_0(R(\varrho)/nR(\varrho) \to G_0(R(\varrho)) \to G_0(R\langle\varrho\rangle)) \to 0$$

so that, for $i=0$, Lenstra's formula takes the simpler form

$$G_0(R\pi) \cong \bigoplus_{\varrho \in X(\pi)} G_0(R\langle\varrho\rangle).$$

But, the strict analogue of this for $i=1$ is not correct.

2. The homomorphism $\varphi : \bigoplus_\varrho G_1(R(\varrho))/H_\varrho \to G_1(R\pi)$

Write $\pi = \prod_p \pi_p$ as the direct product of its p-primary components π_p. For each set S of primes put $\pi_S = \prod_{p \in S} \pi_p$. The canonical homomorphisms $\pi \to \pi_S \hookrightarrow \pi$ induce, by restriction, an exact functor $N_S : \mathcal{M}_{R\pi} \to \mathcal{M}_{R\pi}$.

For $M \in \mathcal{M}_{R\pi}$, $N_S M$ is the R-module M on which π_p acts as given for $p \in S$, and trivially for $p \notin S$. In particular $N_S M = M$ if π_p acts trivially on M for $p \notin S$. Moreover $N_S \cdot N_T = N_{S\cap T}$. We also write

$$N_S : G_i(R\pi) \to G_i(R\pi) \quad (i=0,1)$$

for the homomorphism induced by the functor N_S.

Let $\varrho \in X(\pi)$, $M \in \mathcal{M}_{R(\varrho)}$, and $\alpha \in \mathrm{Aut}_{R(\varrho)}(M)$. We shall write

$$[M,\alpha;(\varrho)] = \text{class of } (M,\alpha) \text{ in } G_1(R(\varrho)),$$
$$[M,\alpha;\langle\varrho\rangle] = \text{class of } (M,\alpha) \text{ in } G_1(R(\varrho))/H_\varrho,$$
$$[M,\alpha;\pi] = \text{class of } (M,\alpha) \text{ in } G_1(R\pi),$$

where we embed $\mathcal{M}_{R(\varrho)}$ in $\mathcal{M}_{R\pi}$ via the canonical projection $R\pi \to R(\varrho)$. Let $P(\varrho)$ denote the set of primes dividing the order of ϱ. We define

$$\varphi'_\varrho : G_1(R(\varrho)) \to G_1(R\pi)$$

by

$$\varphi'_\varrho[M,\alpha;(\varrho)] = \sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} N_S[M,\alpha;\pi].$$

The next lemma will be used to show that $\varphi'_\varrho(H_\varrho)=0$.

Lemma 1. *Let* $M \in \mathcal{M}_{R(\varrho)/nR(\varrho)}$, *where* $\varrho \in X(\pi)$ *has order n. There is a chain of submodules*

$$M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$$

such that, for each i, M_i *is stable under every* $R(\varrho)$*-endomorphism of M, and, for some* $p \in P(\varrho)$, $p(M_i/M_{i+1}) = 0$ *and* ϱ_p *acts trivially on* M_i/M_{i+1}.

Except for the $\mathrm{End}_{R(\varrho)}(M)$-invariance of each M_i, this is Lemma 2.2 of Lenstra [2], and the M_i constructed by Lenstra are clearly $\mathrm{End}_{R(\varrho)}(M)$ invariant.

Corollary. *The subgroup* H_ϱ *of* $G_i(R(\varrho))$ *is generated by elements* $[M, \alpha; (\varrho)]$ *such that, for some prime* $p \in P(\varrho)$, $pM = 0$ *and* ϱ_p *acts trivially on M.*

Now let (M, α) and p be as in the Corollary. Then

$$\varphi'_\varrho[M, \alpha; (\varrho)] = \sum_{p \in S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)}(N_S[M, \alpha; \pi] - N_{S-\{p\}}[M, \alpha; \pi]) = 0.$$

Thus φ'_ϱ induces a homomorphism

$$\varphi_\varrho : G_1(R(\varrho))/H_\varrho \to G_1(R\pi),$$

$$\varphi_\varrho[M, \alpha; \langle\varrho\rangle] = \sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} N_S[M, \alpha; \pi].$$

Let

$$\varphi : \bigoplus_{\varrho \in X(\pi)} G_1(R\pi)/H_\varrho \to G_1(R\pi)$$

be the homomorphism with components $(\varphi_\varrho)_{\varrho \in X(\pi)}$.

Theorem. φ *is an isomorphism.*

3. The inverse $\Psi : G_1(R\pi) \to \bigoplus_\varrho G_1(R(\varrho))/H_\varrho$

Let $\varrho \in X(\pi)$ and let S be a set of primes. The functor $N_S : \mathcal{M}_{R\pi} \to \mathcal{M}_{R\pi}$ carries the subcategory $\mathcal{M}_{R(\varrho)}$ to $\mathcal{M}_{R(\varrho_S)}$, and so defines homomorphism $N_S : G_1(R(\varrho)) \to G_1(R(\varrho_S))$ sending $[M, \alpha; (\varrho)]$ to $[N_S M, \alpha; (\varrho_S)]$. Thus we can define

$$\Psi_\varrho : G_1(R(\varrho)) \to \bigoplus_{\varrho'} G_1(R(\varrho'))/H_\varrho$$

by

$$\Psi_\varrho[M, \alpha; (\varrho)] = \sum_{S \subset P(\varrho)} [N_S M, \alpha; \langle\varrho_S\rangle]. \tag{3.1}$$

Lemma 2. *Suppose* $\varrho_1, \varrho_2 \in X(\pi)$, $M \in \mathcal{M}_{R(\varrho_1)} \cap \mathcal{M}_{R(\varrho_2)}$, and $\alpha \in \mathrm{Aut}_{R\pi}(M)$. *Then*

$$\Psi_{\varrho_1}[M, \alpha; (\varrho_1)] = \Psi_{\varrho_2}[M, \alpha; (\varrho_2)].$$

Lenstra's proof of his analogous Lemma 4.1 applies without any change here.

Lemma 3. *Let $M \in \mathcal{M}_{R\pi}$. There is a chain of submodules*

$$M = M_0 \supset M_1 \supset \cdots \supset M_t = 0 \tag{3.2}$$

such that, for each i, M_i is stable under $\mathrm{End}_{R\pi}(M)$, *and $M_i/M_{i+1} \in \mathcal{M}_{R(\varrho)}$ for some $\varrho \in X(\pi)$.*

Except for the assertion about $\mathrm{End}_{R(\pi)}(M)$-invariance, this is just Lemma 2.5 of Lenstra [2]. The M_i constructed by Lenstra are easily seen to be $\mathrm{End}_{R\pi}(M)$-invariant.

Let $M \in \mathcal{M}_{R\pi}$ and $\alpha \in \mathrm{Aut}_{R\pi}(M)$. With the notation of Lemma 3, let $\alpha_i \in \mathrm{Aut}_{R\pi}(M_i/M_{i+1})$ be the automorphism induced by α, and choose $\varrho_i \in X(\pi)$ so that $M_i/M_{i+1} \in \mathcal{M}_{R(\varrho_i)}$. Put

$$\Psi[M, \alpha; \pi] = \sum_{i=0}^{t-1} \Psi_{\varrho_i}[M_i/M_{i+1}, \alpha_i; \langle \varrho_i \rangle]. \tag{3.3}$$

By Lemma 2, this is independent of the choice of the ϱ_i's. To see that it does not depend on the filtration (2) of M, note first that Ψ is clearly *unaltered* if we replace (3.2) by a refinement. In general any two filtrations of M as in Lemma 3 have refinements which are equivalent in the sense of the Schrier theorem [3, Chapter IV, §4]; Schrier equivalent filtrations of M clearly give the same value for $\Psi[M, \alpha : \pi]$ in (3.1). Finally it is easily seen (as in Lenstra [2, p. 181]) that Ψ respects the defining relations of $G_1(R\pi)$. Thus (3) defines a homomorphism

$$\Psi : G_1(R\pi) \to \bigoplus_{\varrho \in X(\pi)} G_1(R(\varrho))/H_\varrho.$$

To check that $\varphi \circ \Psi[M, \alpha; \pi] = [M, \alpha; \pi]$ it suffices clearly to consider the case when $M \in \mathcal{M}_{R(\varrho)}$ for some $\varrho \in X(\pi)$. Then we have

$$\begin{aligned}
\varphi \circ \Psi[M, \alpha; \pi] &= \varphi\left(\sum_{S \subset P(\varrho)} [N_S M, \alpha; \langle \varrho_S \rangle] \right) \\
&= \sum_{S \subset P(\varrho)} \sum_{T \subset S} (-1)^{\#(S-T)} [N_T N_S M, \alpha; \pi] \\
&= \sum_{T \subset P(\varrho)} [N_T M, \alpha; \pi] \left(\sum_{T \subset S \subset P(\varrho)} (-1)^{\#(S-T)} \right) \\
&= [N_{P(\varrho)} M, \alpha; \pi] = [M, \alpha; \pi].
\end{aligned}$$

On the other hand

$$\begin{aligned}
\Psi \circ \varphi[M, \alpha; \langle \varrho \rangle] &= \Psi\left(\sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} [N_S M, \alpha; \pi] \right) \\
&= \sum_{S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \sum_{T \subset S} [N_T N_S M, \alpha; \langle (\varrho_S)_T \rangle] \\
&= \sum_{T \subset P(\varrho)} [N_T M, \alpha; \langle \varrho_T \rangle] \left(\sum_{T \subset S \subset P(\varrho)} (-1)^{\#(P(\varrho)-S)} \right) \\
&= [N_{P(\varrho)} M, \alpha; \langle \varrho_{P(\varrho)} \rangle] = [M, \alpha; \langle \varrho \rangle].
\end{aligned}$$

This concludes the proof that φ is an isomorphism. One can check, just as in Lenstra [2, Section 5] that φ is functorial with respect to π and R in the same way that Lenstra's isomorphism is for G_0.

References

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